Model Answer

B.Sc (Hon's) Fifth Semester Examination, 2014

Mathematics

Linear Algebra (AU-6866)

1. (i) Since

$$(\bar{1} \oplus \bar{1}) \cdot 1 = \bar{2} \cdot 1 = \bar{0} \cdot 1 = 0$$

$$\bar{1} \cdot 1 + \bar{1} \cdot 1 = 1 + 1 = 2$$

therefore $(\bar{1} + \bar{1}) \cdot 1 \neq \bar{1} \cdot 1 + \bar{1} \cdot 1$

Hence $(R, +, \cdot)$ is not a vector space over $(\mathbb{Z}_2, \oplus, \odot)$.

(ii). A non-empty subset W of a vector space V is said to be subspace of the vector space if W is a vector space over the same field with respect to the induced operations.

(iii). Since $S = \{1, i\}$ is L.I. in $\mathbb{C}$ over $\mathbb{R}$ and $\mathbb{C} = [S]$

therefore S is a basis of $\mathbb{C}(\mathbb{R})$.

(iv). Since $\bar{1} \cdot a \circ \bar{1} \cdot b \circ \bar{1} \cdot c = e$

therefore $\alpha a \circ \beta b \circ \gamma c = e$ has non-zero solution.

Hence $\{a, b, c\}$ is L.D. in $V_4(\mathbb{Z}_2)$.

(v). Let V_1 and V_2 be two vector spaces over the same field F. A map $T: V_1 \to V_2$ is said to linear transformation if $T(\alpha a + \beta b) = \alpha T(a) + \beta T(b) \quad \forall\, a, b \in V_1,\ \alpha, \beta \in F$

eg. $T: V_1 \to V_2$ defined by $T(x) = 0_2 \quad \forall\, x \in V_1$

is a linear transformation.

(vi). Orthogonal set: A subset S of an inner product space is said to be orthogonal set if distinct vectors in S are orthogonal, ie.

$$\langle a_i, a_j \rangle = 0_F \quad \forall\ a_i \neq a_j \in S.$$

~~Orthogonal~~ Orthonormal set: A subset S of an inner product space is said to be orthonormal set if S is a orthogonal set and norm of each vector in S is 1., ie

$$\langle a_i, a_j \rangle = \delta_{ij} = \begin{cases} 1 & ; i = j \\ 0 & ; i \neq j \end{cases}$$

(vii). Cayley-Hamilton Theorem. Every square matrix over a real or complex field satisfies its characteristic equation.

(viii). The dimension of the Image set of a linear transformation is called rank of the linear transformation.

2. Let $S = \{a_1, a_2, \dots a_n\}$ be a non-empty subset of a vector space $V(F)$. Then

$$L(S) = \{ \alpha_1 a_1 + \dots + \alpha_n a_n \ ;\ \alpha's \in F \}$$

and $[S]$ is the smallest subspace of V containg S.

$L(S)$ is a subspace.

Let $\alpha_1 a_1 + \dots + \alpha_l a_l\ ,\ \beta_1 a_1 + \dots + \beta_m a_m \in L(S)$

and $\lambda, \mu \in F$

without loss of generality we may suppose $l < m$. Then

$\lambda \cdot (\alpha_1 a_1 + \dots + \alpha_l a_l) + \mu \cdot (\beta_1 a_1 + \dots + \beta_m a_m)$

$= (\lambda \cdot \alpha_1 + \mu \cdot \beta_1) a_1 + \dots + (\lambda \cdot \alpha_l + \mu \beta_l) a_l + (\mu \beta_{l+1}) a_{l+1} + \dots + (\mu \beta_m) a_m$

$=$ a linear combination of elts of S

$\in L(S)$

L(S) Contains S.

Since $a_1 = 1.a_1 + 0.a_2 + \cdots + 0.a_n \in L(S)$

Illy $a_i = 0.a_1 + 0.a_2 + \cdots + 1.a_i + \cdots 0.a_n \in L(S) \quad \forall i \in \{1, \ldots, n\}$

Hence $S \subseteq L(S)$.

L(S) is the smallest one

Let W be a subspace of $V(F)$ containg S.

then $a_1, a_2, \ldots a_n \in W$

therefore all linear combinations of $a_1, a_2 \ldots a_n$ will be in W

Hence $L(S) \subseteq W$

Therefore $L(S)$ is the smallest subspace containing S.

ie $L(S) = [S]$.

3. Linear sum of two subspaces W_1 and W_2 of the vector space $V(F)$ is defined by

$$W_1 + W_2 \overset{\text{def}}{=} \{ a_1 + a_2 \mid a_1 \in W_1 \ \& \ a_2 \in W_2 \} \subseteq V.$$

A vector space $V(F)$ is said to be direct sum of two subspaces W_1 & W_2 if each element in V can be uniquely written as sum of one element of W_1 and one element of W_2.

$W_1 + W_2$ is a subspace of V.

Let $a_1 + a_2,\ b_1 + b_2 \in W_1 + W_2$ and $\alpha, \beta \in F$

then $\cancel{\alpha(a_1+a_2) + \beta(b_1+b_2)}$

$a_1, b_1 \in W_1$ and $a_2, b_2 \in W_2$

Since W_1 and W_2 are subspace, therefore

$$\alpha a_1 + \beta b_1 \in W_1 \quad \text{and} \quad \alpha a_2 + \beta b_2 \in W_2$$

then $\quad (\alpha a_1 + \beta b_1) + (\alpha a_2 + \beta b_2) \in W_1 + W_2$

ie. $\quad \alpha(a_1 + a_2) + \beta(b_1 + b_2) \in W_1 + W_2$

Hence $W_1 + W_2$ is a subspace of V.

4. Let $X = \{x_1, x_2, ---, x_n\}$ and

$Y = \{y_1, y_2, --- y_m\}$ be two bases of a vector space $V(F)$.

Consider X as a generating subset of V and Y as a L.I. subset of V. We will show that $m \not> n$, ie. $m \leq n$.

Since $y_1 \in V$ and $V = [X]$

$\therefore$ y_1 can be written as a linear combination of $x_1, x_2, ---, x_n$.

then $\{y_1, x_1, x_2, ---, x_n\}$ ~~are~~ is L.D.

~~therefore at least one of $\alpha_1, \alpha_2, \dots \alpha_n$ and β_1 not equal to zero s.t.~~

$$\cancel{\beta_1 y_1 + \alpha_1 x_1 + ---- + \alpha_n x_n = 0}$$

~~We claim that at least one α's is non-zero, otherwise above equation becomes $\beta_1 y_1 = 0$ which implies $y_1 = 0$~~

Therefore $\exists$ an element different from y_1 (say x_i) which is a linear combination of preceding vectors. If we omit this vector x_i from the above set then V is also generated by the remaining set

$$X_1 = \{y_1, x_1, x_2, ---- x_{i-1}, x_{i+1}, --- x_n\}$$

Since $y_2 \in V$ and $V = [X_1]$, therefore y_2 can be expressed as a linear combination of elements of X_1 and hence $\{y_2, y_1, x_1, x_2, --- x_{i-1}, x_{i+1}, --- x_n\}$ is L.D.

Therefore $\exists$ an element of this set different from y_1 and y_2 (as $\{y_1, y_2\}$ is L.I. set) (say x_j) which is a linear combination of preceding vectors. Deleting x_j from above set as before, we get the set

$$X_2 = \{y_2, y_1, x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots x_{j-1}, x_{j+1}, \dots x_n\}$$

which will generate V.

Therefore successive generating sets $X_1, X_2, \dots$ etc. are obtained by excluding an x from X and including a y in X at each step.

Now the set X can not be exhausted before the set Y, (ie. $n \not< m$), otherwise we will get

$$X_n = \{y_n, y_{n-1}, \dots y_2, y_1\} \quad \text{s.t. } V = [X_n]$$

which will be a proper subset of Y and then Y becomes L.D. which is a contradiction.

Hence $n \not< m$ ie. $m \leq n$.

Interchanging the roll of two bases, we can all prove that $n \leq m$.

Therefore we get $m = n$.

Thus the two bases X and Y must have the same number of elements.

5. Let $\dim W = l$ and $S = \{a_1, a_2, \dots a_l\}$ be a basis of W.
Since S is L.I. subset in W, it is L.I. in V and therefore S can be extended to form a basis of V.
Let $S_1 = \{a_1, a_2, \dots a_l, b_1, b_2, \dots b_m\}$ be a basis of V.
then $\dim V = l + m$

Consider $S_2 = \{W+b_1, W+b_2, \cdots + W+b_m\} \subseteq V/W$

We will show that S_2 is a basis of V/W.

S_2 is L.I. in V/W.

Let $\beta_1(W+b_1) + \beta_2(W+b_2) + \cdots + \beta_m(W+b_m) = W$ (zero of V/W) --- (1)

then $W + (\beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m) = W$

which implies $\beta_1 b_1 + \beta_2 b_2 + \cdots + \beta_m b_m \in W$.

therefore $\beta_1 b_1 + \cdots + \beta_m b_m$ can be uniquely written as a linear combination of $a_1, a_2 \ldots a_l$. (being S a basis of W)

ie. $\beta_1 b_1 + \cdots + \beta_m b_m = \alpha_1 a_1 + \cdots + \alpha_l a_l$

ie. $(-\alpha_1) a_1 + \cdots + (-\alpha_l) a_l + \beta_1 b_1 + \cdots + \beta_m b_m = 0$

which implies $\alpha_1 = \cdots = \alpha_l = 0 = \beta_1 = \cdots = \beta_m$, --(2) because $S_1 = \{a_1, a_2 \cdots a_l, b_1, b_2 \cdots b_m\}$ is a L.I. being a basis.

Hence in view of (1) and (2), S_2 is L.I. in V/W.

$V/W = [S_2]$.

Let $W + x \in V/W$ then $x \in V$

therefore $x = \theta_1 a_1 + \cdots + \theta_l a_l + \phi_1 b_1 + \cdots + \phi_m b_m$ as $V = [S_1]$

then $W + x = W + \theta_1 a_1 + \cdots + \theta_l a_l + \phi_1 b_1 + \cdots + \phi_m b_m$

$= W + \phi_1 b_1 + \cdots + \phi_m b_m$ as $\theta_1 a_1 + \cdots + \theta_l a_l \in W$

$= \phi_1 (W + b_1) + \cdots + \phi_m (W + b_m)$

= a linear combination of elements of S_2.

Hence $V/W = [S_2]$.

therefore S_2 is a basis of V/W.

Hence $\dim V/W = m = (l+m) - l$

$= \dim V - \dim W$

6. $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (x+y,\ y+z)$$

$\therefore$ $T(1,1,0) = (1+1,\ 1+0) = (2,1)$

Let $T(1,1,0) = \alpha_1 (2,-3) + \alpha_2 (1,4)$

ie. $(2,1) = (2\alpha_1 + \alpha_2,\ -3\alpha_1 + 4\alpha_2)$

then $\alpha_1 = 7/11$, $\alpha_2 = 8/11$

Similarly $T(1,0,1) = (1+0,\ 0+1) = (1,1)$

Let $T(1,0,1) = \beta_1 (2,-3) + \beta_2 (1,4)$

ie $(1,1) = (2\beta_1 + \beta_2,\ -3\beta_1 + 4\beta_2)$

then $\beta_1 = 3/11$, $\beta_2 = 5/11$

and $T(1,1,-1) = (1+1,\ 1-1) = (2,0)$

Let $T(1,1,-1) = \gamma_1 (2,-3) + \gamma_2 (1,4)$

ie. $(2,0) = (2\gamma_1 + \gamma_2,\ -3\gamma_1 + 4\gamma_2)$

then $\gamma_1 = 8/11$, $\gamma_2 = 6/11$

Therefore

$$mx(T)_{X,Y} = \begin{bmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{bmatrix}$$

$$= \begin{bmatrix} 7/11 & 3/11 & 8/11 \\ 8/11 & 5/11 & 6/11 \end{bmatrix}$$

7. Let V be a vector space over the field F of complex number or real number. A map $\langle\ \rangle : V \times V \to F$ is called an inner product in V if the following conditions are satisfied:

(I) $\langle a+b, c\rangle = \langle a, c\rangle + \langle b, c\rangle$

(II) $\langle \lambda a, b\rangle = \lambda\langle a, b\rangle$ $\quad \forall a, b, c \in V$

(III) $\langle a, b\rangle = \overline{\langle b, a\rangle}$

(iv) $\langle a, a\rangle \geq 0_F$ and $\langle a, a\rangle = 0_F$ iff $a = 0$.

If $\exists$ an inner product $\langle\,\rangle$ in a vector space V then $(V, \langle\,\rangle)$ is called an inner product space.

Let V be an n-dimensional vector space over $\mathbb{C}$. and $S = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis of V.

Let $a, b, c \in V$ s.t.

$$a = \alpha_1 x_1 + \cdots + \alpha_n x_n$$
$$b = \beta_1 x_1 + \cdots + \beta_n x_n$$
$$c = \gamma_1 x_1 + \cdots + \gamma_n x_n$$

Let us define a map $\langle\,\rangle : V \times V \to \mathbb{C}$ by

$$\langle a, b\rangle = \alpha_1\overline{\beta_1} + \alpha_2\overline{\beta_2} + \cdots + \alpha_n\overline{\beta_n} \qquad \forall a, b \in V.$$

we will show that this map $\langle\,\rangle$ is an inner product in V.

(I). Since $a+b = (\alpha_1+\beta_1)x_1 + \cdots + (\alpha_n+\beta_n)x_n$

$$\begin{aligned}
\langle a+b, c\rangle &= (\alpha_1+\beta_1)\overline{\gamma_1} + \cdots + (\alpha_n+\beta_n)\overline{\gamma_n} \\
&= (\alpha_1\overline{\gamma_1} + \beta_1\overline{\gamma_1}) + \cdots + (\alpha_n\overline{\gamma_n} + \beta_n\overline{\gamma_n}) \\
&= (\alpha_1\overline{\gamma_1} + \alpha_2\overline{\gamma_2} + \cdots + \alpha_n\overline{\gamma_n}) + (\beta_1\overline{\gamma_1} + \cdots + \beta_n\overline{\gamma_n}) \\
&= \langle a, c\rangle + \langle b, c\rangle
\end{aligned}$$

(II). Since $\lambda a = \lambda\alpha_1 x_1 + \lambda\alpha_2 x_2 + \cdots + \lambda\alpha_n x_n$

$$\begin{aligned}
\langle \lambda a, b\rangle &= (\lambda\alpha_1)\overline{\beta_1} + \cdots + (\lambda\alpha_n)\overline{\beta_n} \\
&= \lambda\{\alpha_1\overline{\beta_1} + \cdots + \alpha_n\overline{\beta_n}\} \\
&= \lambda\langle a, b\rangle
\end{aligned}$$

(iii). Since $\langle b,a\rangle = \beta_1\bar{\alpha}_1 + \beta_2\bar{\alpha}_2 + \cdots + \beta_n\bar{\alpha}_n$

$$\overline{\langle b,a\rangle} = \overline{\beta_1\bar{\alpha}_1 + \beta_2\bar{\alpha}_2 + \cdots + \beta_n\bar{\alpha}_n}$$

$$= \overline{\beta_1\bar{\alpha}_1} + \cdots + \overline{\beta_n\bar{\alpha}_n}$$

$$= \bar{\beta}_1\alpha_1 + \cdots + \bar{\beta}_n\alpha_n$$

$$= \alpha_1\bar{\beta}_1 + \cdots + \alpha_n\bar{\beta}_n$$

$$= \langle a,b\rangle$$

(iv) $\langle a,a\rangle = \alpha_1\bar{\alpha}_1 + \cdots + \alpha_n\bar{\alpha}_n$

$$= |\alpha_1|^2 + \cdots + |\alpha_n|^2 \geq 0_F$$

and $\langle a,a\rangle = 0_F$

iff $|\alpha_1|^2 + \cdots + |\alpha_n|^2 = 0_F$

iff $|\alpha_1| = |\alpha_2| = \cdots = |\alpha_n| = 0_F$

iff $a = 0$.

Hence the map $\langle\,\rangle$ is an inner product and $(V, \langle\,\rangle)$ is an inner product space.

Q. First define the orthogonal complement of a subspace.

Let $S = \{a_1, a_2, \ldots a_m\}$ be an orthonormal basis of W.

then $\dim W = m$.

The set S can be extended to an orthonormal basis of the inner product space V.

Let $S_1 = \{a_1, a_2, \ldots a_m, b_1, b_2, \ldots b_n\}$ be an orthonormal basis of V.

then $\dim V = m+n$.

Consider $S_2 = \{b_1, b_2, \ldots b_n\}$.

We will show that S_2 is an orthonormal basis of $W^\perp$.

S_2 is a subset of $W^\perp$.

Since S is an orthonormal set therefore

$b_1, b_2 \ldots b_n$ are orthogonal to $a_1, a_2, \ldots a_m$

and then $b_1, b_2 \ldots b_n$ are orthogonal to every linear Combination of $a_1, a_2, \ldots a_m$.

i.e. $b_1, b_2, \ldots b_n$ are orthogonal to all elements of W.

therefore $S_2 \subseteq W^{\perp}$.

<u>S_2 is L.I.</u>

Since $S_2 \subseteq S_1$ and S_1 is L.I. therefore S_2 is L.I.

<u>$W^{\perp} = [S_2]$.</u>

Let $x \in W^{\perp}$ then $x \in V$

therefore $x = \langle x, a_1 \rangle a_1 + \langle x, a_2 \rangle a_2 + \cdots + \langle x, a_m \rangle a_m$

$+ \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_n \rangle b_n$

(as $S_1 = \{a_1 \ldots a_m, b_1 \ldots b_n\}$ is orthonormal basis of V)

$x = \langle x, b_1 \rangle b_1 + \cdots + \langle x, b_n \rangle b_n$ as $\langle x, a_i \rangle = 0_F$ being $x \in W^{\perp}$ and $a_i \in W$

therefore $W^{\perp} = [S_2]$

Hence S_2 is a basis of $W^{\perp}$.

We will now show that $V = W + W^{\perp}$.

Let $y \in V$ then

$y = \langle y, a_1 \rangle a_1 + \cdots + \langle y, a_m \rangle a_m$

$+ \langle y, b_1 \rangle b_1 + \cdots + \langle y, b_n \rangle b_n$

then $y = w + v$ where $w = \langle y, a_1 \rangle a_1 + \cdots + \langle y, a_m \rangle a_m \in W$

and $v = \langle y, b_1 \rangle b_1 + \cdots + \langle y, b_n \rangle b_n \in W^{\perp}$

Hence $V = W + W^{\perp}$. ---(1)

Also $W \cap W^{\perp} = \{0\}$, because if $x \in W \cap W^{\perp}$ then $x \in W$ and $x \in W^{\perp}$ --(2) then $\langle x, x \rangle = 0_F$ and therefore $x = 0$.

From (1) & (2), we get

$V = W \oplus W^{\perp}$.